Cateogry

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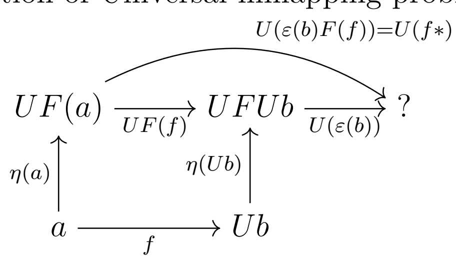
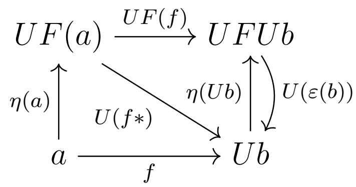
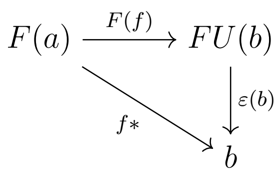
1 Adjunction to Universal mapping

In adjunction $(F, U, \varepsilon, \eta)$, put $f* = \varepsilon(b)F(f)$, we are going to prove $f*$ is a solution of Universal mmapping problem. That is $U(f*)\eta = f$.







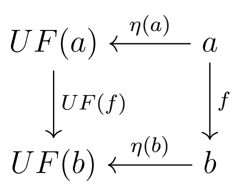
$\because U\varepsilon \circ \eta U = 1_U$

$$U(\varepsilon(b))\eta(U(b)) = 1_{U(b)}$$

means that
$\varepsilon(b) : FU(b) \to b$ is solution of $1_{U(b)}$.
naturality $f\eta(U(b)) = U(F(f))\eta(a)$
gives solution $U(\varepsilon(b))UF(f) = U(F(f)\varepsilon(b))$ for f.

$$U(f*)\eta(a)(a) = f(a)$$

then
$U(\varepsilon(b)F(f))\eta(a)(a) = U(\varepsilon(b))UF(f)\eta(a)(a)$
since F is a functor. And we have
$U(\varepsilon(b))UF(f)\eta(a)(a) = U(\varepsilon(b))\eta(b)f(a)$
because of naturality of η



$UF(f)\eta(a) = \eta(b)f$

too bad.... we need some thing more.

1.1 Adjoint of η

$U\varepsilon \circ \eta U = 1_U$

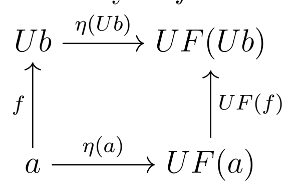
$$F(a) \xrightarrow{F(f)} FU(b) \xrightarrow{\varepsilon(b)} b$$

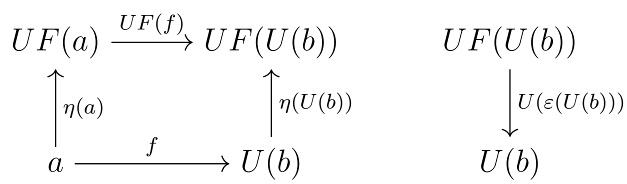
$$UF(a) \xrightarrow{UF(f)} UFU(b) \xrightarrow{U(\varepsilon(b))} U(b)$$

$$a \xrightarrow{\eta(a)} UF(a) \xrightarrow{UF(f)} UFU(b) \xrightarrow{U(\varepsilon(b))} U(b)$$

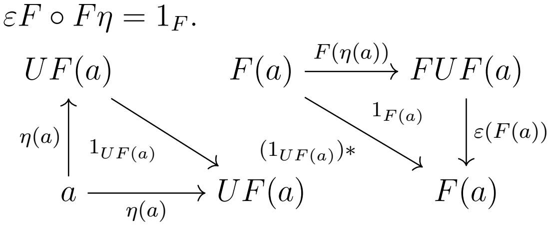
$$a \xrightarrow{f} Ub \xrightarrow[\eta(Ub)]{\eta(Ub)} UFU(b) \xrightarrow[U(\varepsilon(b))=1]{U(\varepsilon(b))} U(b)$$

$\because U\varepsilon \circ \eta U = 1_U$
naturality of $f : a \to Ub$





Solution of universal mapping yields naturality of $U\varepsilon \circ \eta U = 1_U$.
$\varepsilon F \circ F\eta = 1_F$.



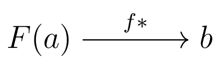
2 Universal mapping to adjunction

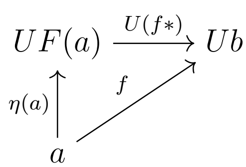
Functor U, mapping $F(a)$ and $(f)*, U(f*)\eta(a) = f$ are given.
object $F(a) : A \to B$
$\eta(a) : a \to UF(a)$
put

$$F(f) = (\eta(b)f)*$$

$$\varepsilon : FU \to 1_B$$

$$\varepsilon(b) = (1_{U(b)})*$$





$f = U(f*)\eta$
Show F is a Functor, that is $F(fg) = F(f)F(g)$.
Show naturality of $\eta(a)$.

$$f : a \to b, F(f) = (\eta(b)f)*$$

Show naturality of $\varepsilon(b) = (1_U)*$.

2.1 Definitions

f's destination

$$f : a \to U(b)$$

universal mapping

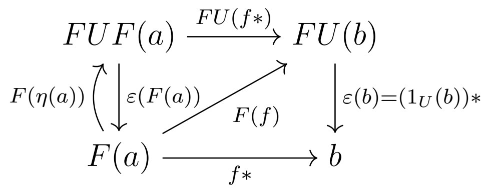
$$U(f*)\eta(a) = f$$

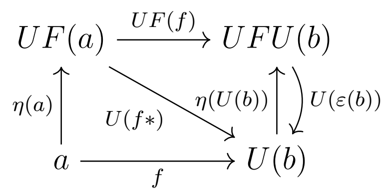
defnition of F(f)

$$F(f) = (\eta(U(b))f)*$$

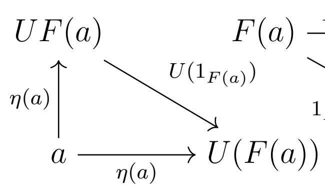
definition of ε

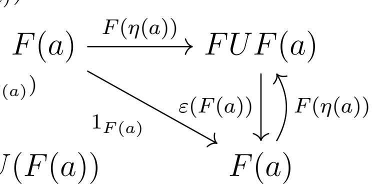
$$\varepsilon(b) = (1_{U(b)})*$$





$\varepsilon F \circ F\eta = 1_F$, $\varepsilon(b) = (1_{U(b)})*$,
$\varepsilon(F(a)) = (1_{UF(a)})*$





2.2 Functor F

$$F(f) = (\eta(b)f)*$$

$$U(F(f))\eta(a) = \eta(b)f$$

show $F(fg) = F(f)F(g)$
$a \xrightarrow{g} Ub \xrightarrow{f} UUc$

$$\begin{aligned} U(F(g))\eta(a) &= \eta(Ub)g \\ U(F(f))\eta(Ub) &= \eta(UUc)f \end{aligned}$$

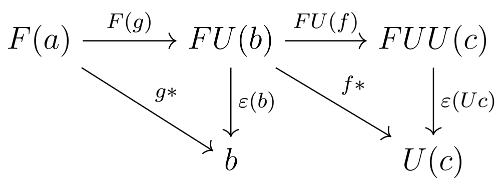
show

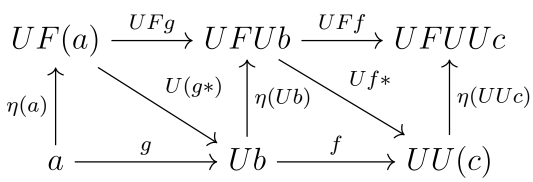
$$U(F(f)F(g))\eta(a) = \eta(UUc)fg$$

then $F(f)F(g) = F(fg)$

$$\begin{aligned} U(F(f)F(g))\eta(a) &= UF(f)UF(g)\eta(a) \\ &= UF(f)\eta(Ub)g \\ &= \eta(UUc)fg \end{aligned}$$

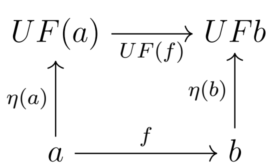
Q.E.D.





2.3 naturality of η

$\eta : 1 \to UB$



prove $\eta(b)f = UF(f)\eta(a)$

$$\begin{array}{rcll} \eta(b)f: & a \to UFb & & \\ F(f) & = & (\eta(b)f)* & \text{(definition)} \\ \eta(b)f & = & U(F(f))\eta(a) & \end{array}$$

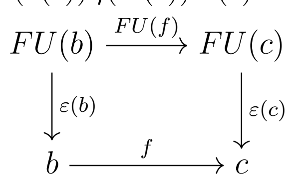
Q.E.D.

2.4 naturality of ε

$$\varepsilon : FU \to 1_B$$

$$U : B \to A$$

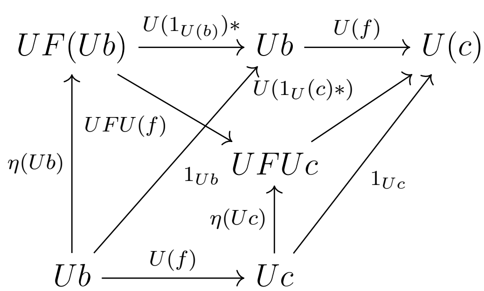
$\varepsilon(b) = (1_{U(b)})*$
$U(\varepsilon(b))\eta(U(b)) = 1_{U(b)}$
$U(\varepsilon(b))\eta(U(b))U(b) = U(b)$



prove $f\varepsilon(b) = \varepsilon(c)FU(f)$

$$f = Ub \to Uc$$

$$F(Ub) \xrightarrow{(1_{U(b)})*} b \xrightarrow{f} c$$



$$F(Ub) \xrightarrow{FU(f)} FU(c) \xrightarrow{(1_{U(c)})*} c$$

show $\varepsilon(c)FU(f)$ and $f\varepsilon(b)$ are both solution of $(1_{Uc})U(f)(= U(f)(1_{Ub}))$

$$(f\varepsilon(b)))\eta(Ub)Ub = U(f))U(\varepsilon(b))\eta(Ub)Ub$$

$$= U(f)1_{U(b)}Ub = U(f)Ub = Ufb = U(f)(1_{Ub})Ub$$

$$\therefore f\varepsilon(b) = (U(f)(1_{Ub}))*$$

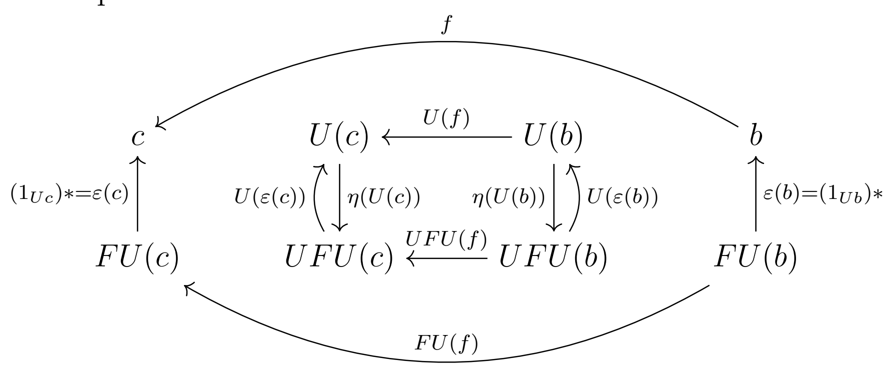
$UFU(f)\eta(Ub) = \eta(Uc)U(f)$ naturality of η

$$U(\varepsilon(c)FU(f))\eta(Ub)Ub = U(\varepsilon(c))UFU(f)\eta(Ub)Ub$$

$$= U(\varepsilon(c))\eta(Uc)U(f)Ub = 1_{U(c)}U(f)Ub = U(f)Ub = U(f)(1_{Ub})Ub$$

$$\because U(\varepsilon(c))\eta(Uc) = 1_U(c)$$

end of proof.



It also prove

$$U\varepsilon \circ \eta U = 1_U$$

2.5 $U\varepsilon \circ \eta U = 1_U$

$\varepsilon(b) = (1_U(b))*$
that is
$U((1_U(b)*)\eta(U(b)) = 1_U(b)$ $U(\varepsilon(b))\eta(U(b)) = 1_U(b)$
$\therefore U\varepsilon \circ \eta U = 1_U$

2.6 $\varepsilon F \circ F\eta = 1_F$

$\eta(a) = U(1_F(a))\eta(a)$
$\Rightarrow (\eta(a))* = 1_F(a)$ ... (1)
$\varepsilon(F(a)) = (1_U F(a))*$
$\Rightarrow 1_U F(a) = U(\varepsilon(F(a)))\eta(UF(a))$
times $\eta(a)$ from left
$\eta(a) = U(\varepsilon(F(a)))\eta(UF(a))\eta(a)$

$\eta(UF(a)) = UF\eta(a)$ naturality of η

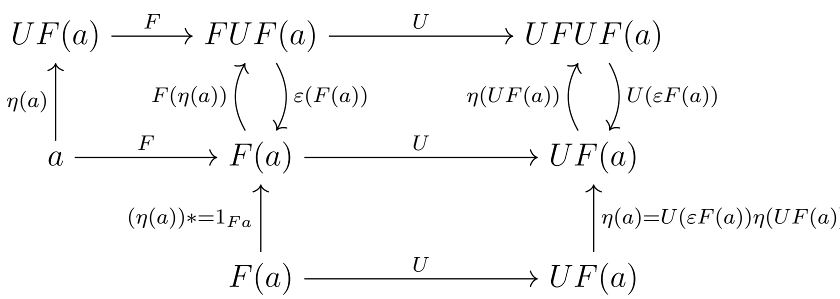
$\eta(a) = U(\varepsilon(F(a)))(UF\eta(a))\eta(a)$

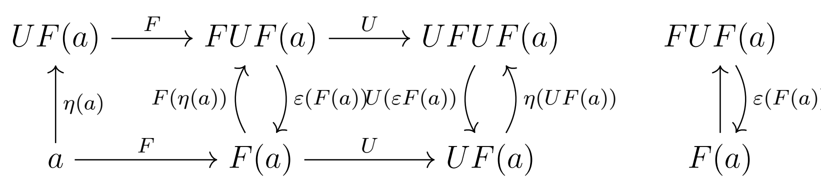
$= U(\varepsilon(F(a)F\eta(a)))\eta(a)$

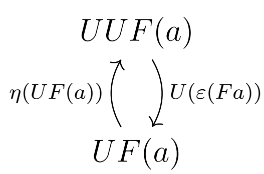
$\Rightarrow (\eta(a))* = \varepsilon(F(a))F\eta(a)....(2)$

from (1),(2), since $(\eta(a))*$ is unique

$$\varepsilon(F(a))F\eta(a) = 1_F(a)$$

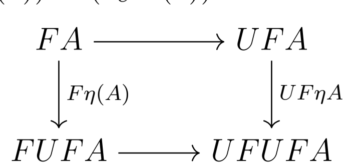






$U(\varepsilon(F(a)))\eta(UF(a)) = 1_U F(a)$

$\varepsilon(F(a)) = (1_U F(a))*$



$\varepsilon(FA)$ の定義から $U(\varepsilon(FA)) : UFUFA \to UFA$

唯一性から $\varepsilon(F(A)) : FUFA \to FA$ 従って

$$\varepsilon(F(A))F\eta(A) = 1$$

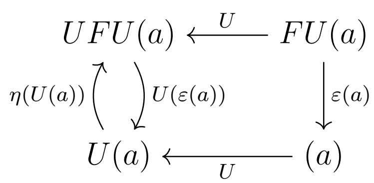
ってなのを考えました。

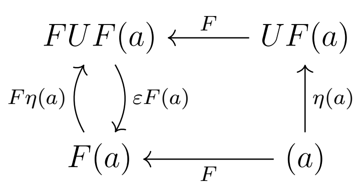
$U\eta(A') = U(1(FA'))\eta(A')$ より
$\eta(A')* = 1(FA')$、
$U\eta(A') = U(\varepsilon(FA')F\eta(A'))\eta(A')$ より
$\eta(A')* = \varepsilon(FA')F\eta(A')$ から
$1_F = \varepsilon F.F\eta$ は言えました。

後者で η の自然性と ε の定義を使いました。

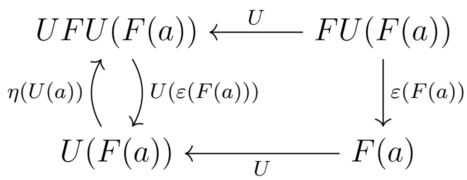
3 おまけ

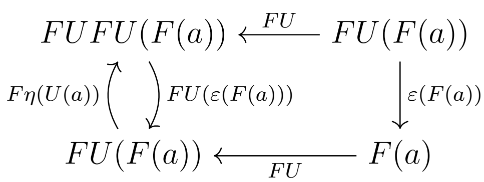
$\varepsilon F \circ F\eta = 1_F, U\varepsilon \circ \eta U = 1_U$

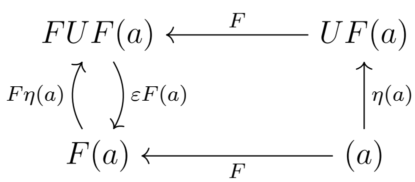




なら、 $FU(\varepsilon(F(a))) = \varepsilon F(a)$?







4 Monad

(T, η, μ)

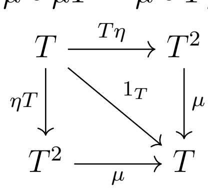
$T : A \to A$

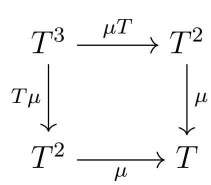
$\eta : 1_A \to T$

$\mu : T^2 \to T$

$\mu \circ T\eta = 1_T = \mu \circ \eta T$ Unity law

$\mu \circ \mu T = \mu \circ T\mu$ association law





5 Adjoint to Monad

Monad (UF, η, U ε F) on adjoint (U,F, η, ε)

$$\begin{aligned} \varepsilon F \circ F\mu &= 1_F \\ U\varepsilon \circ \mu U &= 1_U \end{aligned}$$

$$\begin{aligned} \mu \circ T\eta &= (U\varepsilon F) \circ (UF\eta) = U(\varepsilon F \circ F\eta) = U1_F = 1_{UF} \\ \mu \circ \eta T &= (U\varepsilon F) \circ (\eta UF) = (U\varepsilon \circ \eta U)F = 1_U F = 1_{UF} \end{aligned}$$

$$\begin{aligned} (U\varepsilon F) \circ (\eta UF) &= (U(\varepsilon(F(b))))(UF(\eta(b))) \\ &= U(\varepsilon(F(b))F(\eta(b))) = U(1_F) \end{aligned}$$

$$\begin{CD} UFUFUF @>{U\varepsilon FUF}>> UFUF \\ @V{UFU\varepsilon F}VV @VV{U\varepsilon F}V \\ UFUF @>{U\varepsilon F}>> UF \end{CD} \qquad \begin{CD} FUFUF @>{\varepsilon FUF}>> FUF \\ @V{FU\varepsilon F}VV @VV{\varepsilon F}V \\ FUF @>>{\varepsilon F}> F \end{CD} \qquad \begin{CD} FU(a) @>{\varepsilon(a)}>> a \\ @V{FU(f)}VV @VV{f}V \\ FU(b) @>>{\varepsilon(b)}> b \end{CD}$$

association law

$$\begin{aligned} \mu \circ \mu T &= \mu \circ T\mu \\ U\varepsilon(a)F \circ U\varepsilon(a)FFU &= U\varepsilon(a)F \circ FUU\varepsilon(a)F \end{aligned}$$

$U\varepsilon F \circ U\varepsilon FFU = U\varepsilon F \circ FUU\varepsilon F$
naturality of ε
$\varepsilon(b)FU(f)(a) = f\varepsilon(a)$

$a = FUF(a), b = F(a), f = \varepsilon F$

$$
\begin{aligned}
\varepsilon(F(a))(FU(\varepsilon F))(a) &= (\varepsilon F)(\varepsilon FUF(a)) \\
U(\varepsilon(F(a))(FU(\varepsilon F))(a)) &= U((\varepsilon F)(\varepsilon FUF(a))) \\
U(\varepsilon(F(a))(FU(\varepsilon F))(a)) &= U((\varepsilon F)(\varepsilon FUF(a))) \\
U\varepsilon F \circ FUU\varepsilon F &= U\varepsilon F \circ U\varepsilon FFU
\end{aligned}
$$

$$
\begin{CD}
FUF(a) @<{FU(\varepsilon(F(a)))}<< FUFUF(a) \\
@VV{\varepsilon(F(a))}V @VV{\varepsilon(FUF(a))}V \\
F(a) @<<{\varepsilon(F(a))}< FUF(a)
\end{CD}
$$

6 Eilenberg-Moore category

(T, η, μ)

A^T object (A, ϕ)

$\phi\eta(A) = 1_A, \phi\mu(A) = \phi T(\phi)$

arrow f.

$\phi T(f) = f\phi$

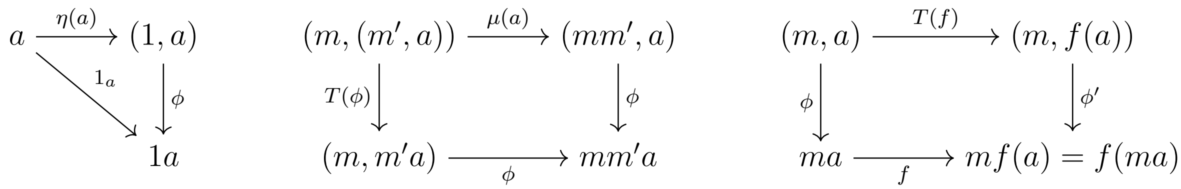
$$
\begin{tikzcd}
a \arrow[r, "\eta(a)"] \arrow[dr, "1_a"'] & T(a) \arrow[d, "\phi"] \\
& a
\end{tikzcd}
\qquad
\begin{CD}
T^2(a) @>{\mu(a)}>> T(a) \\
@V{T(\phi)}VV @VV{\phi}V \\
T(a) @>>{\phi}> T(a)
\end{CD}
\qquad
\begin{CD}
T(a) @>{T(f)}>> T(b) \\
@V{\phi}VV @VV{\phi'}V \\
a @>>{f}> b
\end{CD}
$$

7 EM on monoid

$f : a \to b$

$T : a \to (m, a)$

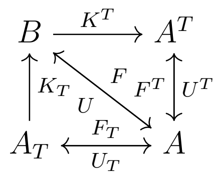
$\eta : a \to (1, a)$
$\mu : (m, (m', a)) \to (mm', a)$
$\phi : (m, a) \to \phi(m, a) = ma$

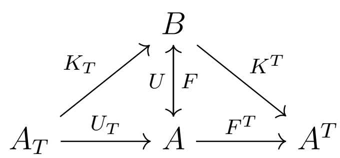


object (a, ϕ). arrow f.
$\phi T(f)(m, a) = f\phi(m, a)$
$\phi(m, f(a)) = f(a)$
$U^T : A^T \to A$
$U^T(a, \phi) = a, U^T(f) = f$
$F^T : A \to A^T$
$F^T(a) = (T(a), \mu(a)), F^T(f) = T(F)$

8 Comparison Functor K^T

$K^T(B) = (U(B), U\varepsilon(B))a, K^T(f) = U(g)$
$U^T K^T(b) = U(b)$
$U^T K^T(f) = U^T U(f) = U(f)$
$K^T F(a) = (UF(a), U\varepsilon(F(a))) = (T(a), \mu(a)) = F^T(a)$
$K^T F(f) = UF(f) = T(f) = F^T(f)$
$\eta U(a, \phi) = \eta(a), U\varepsilon(a, \phi) = \varepsilon^T K^T(b) = U\varepsilon(b)$





9 Kleisli Category

Object of A.

Arrow $f : a \to T(a)$ in A. In A_T, $f : b \to c, g : c \to d$,
$g * f = \mu(d)T(g)f$
$\eta(b) : b \to T(b)$ is an identity.
$f * \eta(b) = \mu(c)T(f)\eta(b) = \mu(c)\eta(T(c))f = 1_T(c)f = f$
and
$\eta(c) * f = \mu(c)T\eta(c)f = 1_T(c)f = f$
association law $g * (f * h) = (g * f) * h$,
$h : a \to T(b), f : b \to T(c), g : c \to T(d)$,
naturality of μ

$$f * h \qquad T(c) \xleftarrow{\mu(c)} T^2(c) \xleftarrow{T(f)} T(b) \xleftarrow{h} a$$

$$g * (f * h) \qquad T(d) \xleftarrow{\mu(d)} T^2(d) \xleftarrow{T(g)} T(c) \xleftarrow{\mu(c)} T^2(c) \xleftarrow{T(f)} T(b) \xleftarrow{h} a$$

$$(g * f) * h \qquad T(d) \xleftarrow{\mu(d)} T^2(d) \xleftarrow{\mu(d)T} T^3(d) \xleftarrow{T^2(g)} T^2(c) \xleftarrow{T(f)} T(b) \xleftarrow{h} a$$

$$(g * f) * h \qquad T(d) \xleftarrow{\mu(d)} T^2(d) \xleftarrow{T\mu(d)} T^3(d) \xleftarrow{T^2(g)} T^2(c) \xleftarrow{T(f)} T(b) \xleftarrow{h} a$$

$$g * f \qquad T(d) \xleftarrow{\mu(d)} T^2(d) \xleftarrow{T(g)} T(c) \xleftarrow{f} b$$

$$\begin{array}{ccccc}
T^2(d) & \xleftarrow{T\mu(d)} & T^2(T(d)) & \xleftarrow{T^2(g)} & T^2(c) \\
\downarrow{\scriptstyle \mu(d)} & & \downarrow{\scriptstyle \mu(T(d))} & & \downarrow{\scriptstyle \mu(c)} \\
T(d) & \xleftarrow{\mu(d)} & T(T(d)) & \xleftarrow{T(g)} & T(c)
\end{array}$$

$$\begin{aligned}
g * (f * h) &= g * (\mu(c)T(f)h) \\
&= \mu(d)(T(g))(\mu(c)T(f)h) \\
&= \mu(d)T(g)\mu(c)T(f)h
\end{aligned}$$

$$\begin{aligned}
(g * f) * h &= (\mu(d)T(g)f) * h \\
&= \mu(d)T(\mu(d)T(g)f)h \\
&= \mu(d)T(\mu(d))T^2(g)T(f)h
\end{aligned}$$

$\mu(d)\mu(d)T = \mu(d)T\mu(d)$
$\mu(T(d))T^2(g) = T(g)\mu(c)$ naturality of μ.
$\mu(d)T\mu(d)T^2(g) = \mu(d)\mu(T(d))T^2(g) = \mu(d)T(g)\mu(c)$

$$
\begin{array}{ccc}
T^3(d) & \xleftarrow{T^2(g)} & T^2(c) \\
\downarrow{\scriptstyle \mu(T(d))} & & {\scriptstyle \mu(c)}\uparrow \\
T^2(d) & \xleftarrow[T(g)]{} & T(c)
\end{array}
$$

$\mu(T(d)) = T\mu(d)$?
$(m, (m'm'', a)) = (mm', (m'', a))$ No, but
$\mu\mu(T(d)) = \mu T\mu(d)$.

10 Ok

$T(g)\mu(c) = T(\mu(d))T^2(g)$ であれば良いが。
$\mu(d)T^2(g) = T(g)\mu(c)$
ちょっと違う。 $\mu(d)T(\mu(d))T^2(g)$ が、
$\mu(d)\mu(d)T^2(g)$
となると良いが。
$\mu(d)T(\mu(d)) = \mu(d)\mu(T(d))$

11 monoid in Kleisli category

$T : a \to (m, a)$
$T : f \to ((m, a) \to (m, f(a)))$
$\mu(a) : (m, (m', a)) \to (mm', a)$
$f : a \to (m, f(a))$

$$
\begin{aligned}
g * f(b) &= \mu(d)T(g)f(b) = \mu(d)T(g)(m, f(b)) \\
&= \mu(m, (m', gf(b))) = (mm', gf(b)) \\
(g * f) * h(a) &= \mu(d)T(\mu(d)T(g)f)h(a) = \mu(d)T(\mu(d))(TT(g))T(f)(m, h(a)) \\
&= \mu(d)T(\mu(d))(TT(g))(m, (m', fh(a))) \\
&= \mu(d)T(\mu(d)(m, (m', (m'', gfh(a))))) = (mm'm'', gfh(a)) \\
g * (f * h)(a) &= (\mu(d)(T(g)))\mu(c)T(f)h(a) = (\mu(d)(T(g)))\mu(c)T(f)(m, h(a)) \\
&= (\mu(d)(T(g)))\mu(c)(m, (m', fh(a)))
\end{aligned}
$$

$$= \mu(d)T(g)(mm', fh(a)) = (mm'm'', gfh(a))$$

12 Resolution of Kleiseli category

$f : a \to b, g : b \to c$

$U_T : A_T \to A$
$U_T(a) = T(a)$
$U_T(f) = \mu(b)T(f)$
$g * f = \mu(d)T(g)f$

$$\begin{aligned} U_T(g * f) &= U_T(\mu(c)T(g)f) \\ &= \mu(c)T(\mu(c)T(g)f) \\ &= \mu(c)\mu(c)T(T(g)f)) = \mu(c)\mu(c)TT(g)T(f) \text{ association law} \\ U_T(g)U_T(f) &= \mu(c)T(g)\mu(b)T(f) = \mu(c)\mu(c)TT(g)T(f) \\ T(g)\mu(b) &= \mu(c)TT(g) \end{aligned}$$

$$\begin{array}{ccc} TT & \xleftarrow{TT(g)} & TT \\ \downarrow{\scriptstyle \mu(c)} & & \downarrow{\scriptstyle \mu(b)} \\ T & \xleftarrow{T(g)} & T \end{array}$$

$F_T : A \to A_T$
$F_T(a) = a$
$F_T(f) = \eta(b)f$
$F_T(1_a) = \eta(a) = 1_{F_T(a)}$

$$\begin{aligned} F_T(g) * F_T(f) &= \mu(c)T(F_T(g))F_T(f) \\ &= \mu(c)T(\eta(c)g)\eta(b)f \\ &= \mu(c)T(\eta(c))T(g)\eta(b)f \\ &= T(g)\eta(b)f \text{ unity law} \\ &= \eta(c)gf = F_T(gf) \end{aligned}$$

$\eta(c)g = T(g)\eta(b)$

$$\begin{array}{ccc} c & \xleftarrow{g} & b \\ \downarrow{\scriptstyle \eta(c)} & & \downarrow{\scriptstyle \eta(b)} \\ T & \xleftarrow{T(g)} & T \end{array}$$

$\mu \circ T\eta = 1_T = \mu \circ \eta T$ Unity law

$$\varepsilon_T(a) = 1_{T(a)}$$

$$U_T \varepsilon_T F_T = \mu$$

$$U_T \varepsilon_T F_T a(a) = U_T \varepsilon_T(a) = U_T(1_{T(a)} = \mu(a)$$

$$\begin{aligned} \varepsilon_T(F_T(a)) * F_T(\eta(a)) &= \varepsilon_T(a) * F_T(\eta(a)) \\ &= 1_{T(a)} * (F_T(\eta(a))) \\ &= 1_{T(a)} * (\eta(T(a))\eta(a))) \\ &= \mu(T(a))T(1_{T(a)})(\eta(T(a))\eta(a))) \\ &= \mu(T(a))\eta(T(a))\eta(a) \\ &= \eta(a) = 1_{F_T} \end{aligned}$$

$$\begin{aligned} U_T(\varepsilon_T(a))\eta(U_T(a)) &= U_T(1_{T(a)}\eta(T(a))) \\ &= \mu(T(a))T(1_{T(a)})\eta(T(a)) \\ &= \mu(T(a))\eta(T(a))1_{T(a)} \\ &= 1_{T(a)} = T(1_a) = 1_{U_T} \end{aligned}$$

12.1 Comparison functor K_T

Adjoint (B, U, F, ε), $K_T : A_T \to B$,
$g : b \to c$.

$$\begin{array}{rcl}
K_T(a) &=& F(a) \\
K_T(g) &=& \varepsilon(F(c))F(g) \\
K_T F_T(a) &=& K_T(a) = F(a) \\
K_T F_T(f) &=& K_T(\eta(b)f) \\
&=& \varepsilon(F(b))F(\mu(b)f) \\
&=& \varepsilon(F(b))F(\mu(b))F(f) = F(f)
\end{array}$$

$$\begin{array}{rcl}
K_T(\eta(b)) &=& \varepsilon(F(b))F(\eta(b)) = 1_{F(b)} \\
K_T(\eta(T(c))g) &=& \varepsilon(F(T(c)))F(\eta(T(c))g) = F(g) \\
K_T(g)K_T(f) &=& \varepsilon(F(c))F(g)\varepsilon(F(b))F(f) = \varepsilon(F(c))\varepsilon(F(c))FUF(g)F(f) \\
K_T(g * f) &=& \varepsilon(F(c))F(\mu(c)UF(g)f) = \varepsilon(F(c))F(\mu(c))FUF(g)F(f) \\
\varepsilon(F(c))FUF(g) &=& F(g)\varepsilon(F(b))
\end{array}$$

$$\begin{array}{ccc}
FU(F(c)) & \xleftarrow{FU(F(g))} & FU(F(b)) \\
\downarrow{\scriptstyle \varepsilon(F(c))} & & \downarrow{\scriptstyle \varepsilon(F(b))} \\
F(c) & \xleftarrow{\quad F(g) \quad} & F(b)
\end{array}$$

$\varepsilon(F(c))F(\mu(c)) = \varepsilon(F(c))\varepsilon(F(c))$?
$\varepsilon(F(c))F(\mu(c)) = \varepsilon(F(c))FU\varepsilon(F(c))$

$$\begin{array}{ccc} FUFU(c) & \xleftarrow{FU\varepsilon(F(c))} & FUFU(F(c)) \\ \Big\downarrow{\scriptstyle \varepsilon F(c))} & & \Big\downarrow{\scriptstyle \varepsilon(F(c))} \\ FU(c) & \xleftarrow{\varepsilon(F(c))} & FU(F(c)) \end{array}$$

$UK_T(a) = UF(a) = T(a) = U_T(a)$
$UK_T(g) = U(\varepsilon((F(c))F(g))) = U(\varepsilon(F(c)))UF(g) = \mu(c)T(g) = U_T(g)$

13 Monoid

$T : A \rightarrow MxA$
$T(a) = (m, a)$
$T(f) : T(A) \rightarrow T(f(A))$
$T(f)(m, a) = (m, f(a))$
$T(fg)(m, a) = (m, fg(a))$

14 association of Functor

$T(f)T(g)(m, a) = T(f)(m, g(a)) = (m, fg(a)) = T(fg)(m, a)$
$\mu : TxT \rightarrow T$
$\mu_a(T(T(a)) = \mu_A((m, (m', a))) = (m * m', a)$

15 TT

$TT(a) = (m, (m', a))$
$TT(f)(m, (m', a)) = (m, (m', f(a))$

16 naturality of μ

$$
\begin{array}{ccc}
TT(a) & \xrightarrow{\mu(a)} & T(a) \\
\downarrow{\scriptstyle TT(f)} & & \downarrow{\scriptstyle T(f)} \\
TT(b) & \xrightarrow{\mu(b)} & T(b)
\end{array}
$$

$\mu(b)TT(f)TT(a) = T(f)\mu(a)TT(a)$
$\mu(b)TT(f)TT(a) = \mu(b)((m, (m', f(a))) = (m * m', f(a))$
$T(f)\mu(a)(TT(a)) = T(f)(m * m', a) = (m * m', f(a))$

17 $\mu \bigcirc \mu$

Horizontal composition of μ
$f \to \mu_T T(a)$
$a \to TT(a)$
$\mu_T(a)TTT(a) = \mu_T(a)(m, (m', (m'', a))) = (m * m', (m'', a))$

$$
\begin{array}{ccc}
TTTT(a) & \xrightarrow{\mu(TTT(a))} & TTT(a) \\
{\scriptstyle TT(\mu)}\downarrow & & \downarrow{\scriptstyle T(\mu)} \\
TTT(a) & \xrightarrow{\mu(TT(a))} & TT(a)
\end{array}
$$

$$
\begin{aligned}
T(\mu_a)\mu_a TTTT(a) &= T(\mu_a)\mu_a(m_0, (m_1, (m_2, (m_3, a))))) \\
&= T(\mu_a)(m_0 * m_1, (m_2, (m_3, a))) = (m_0 * m_1, (m_2 * m_3, a)) \\
\mu_b TT(\mu_a)TTTT(a) &= \mu_b TT(\mu_a)(m_0, (m_1, (m_2, (m_3, a))))) \\
&= \mu_b(m_0, (m_1, (m_2 * m_3, a))) = (m_0 * m_1, (m_2 * m_3, a))
\end{aligned}
$$

Horizontal composition of natural transformation

18 Natural transformation ε and Functor $F : A \to B, U : B \to A$

$\varepsilon : FUFU \to FU$

$\varepsilon : FU \to 1_B$

Naturality of ε

$$\begin{array}{ccc} FU(a) & \xrightarrow{\varepsilon(a)} & a \\ \downarrow{\scriptstyle FU(f)} & & \downarrow{\scriptstyle f} \\ FU(b) & \xrightarrow{\varepsilon(b)} & b \end{array}$$

$$\varepsilon(b)FU(f)a = f\varepsilon(a)a$$

$$\begin{array}{ccc} FUFU(a) & \xrightarrow{\varepsilon(FU(a))} & FU(a) \\ \downarrow{\scriptstyle FUFU(f)} & & \downarrow{\scriptstyle FU(f)} \\ FUFU(b) & \xrightarrow{\varepsilon(FU(b))} & FU(b) \end{array}$$

$$\varepsilon((FU(b))FUFU(f)FU(a) = FU(f)\varepsilon(FU(a))FU(a)$$

19 Vertcial Compositon $\varepsilon \bullet \varepsilon$

$\varepsilon \bullet \varepsilon : FUFU \to 1B$

$$\begin{array}{ccccc} FUFU(a) & \xrightarrow{\varepsilon(FU(a))} & FU(a) & \xrightarrow{\varepsilon(a)} & a \\ \downarrow{\scriptstyle FUFU(f)} & & \downarrow{\scriptstyle FU(f)} & & \downarrow{\scriptstyle f} \\ FUFU(b) & \xrightarrow{\varepsilon(FU(b))} & FU(b) & \xrightarrow{\varepsilon(b)} & b \end{array}$$

20 Horizontal Composition $\varepsilon \circ \varepsilon$

$$FUFU \longleftarrow FU \longleftarrow B$$

$$\begin{array}{ccc} FU & \qquad & FU \\ \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon} \\ 1_B & & 1_B \end{array}$$

$$B \longleftarrow B \longleftarrow B$$

cf. $FUFU, FU$ has objects of B.

$\varepsilon \circ \varepsilon : FUFU \to 1_B 1_B$

$$\begin{array}{ccc} FUFU(b) & \xrightarrow{\varepsilon FU(b)} & 1_A \\ \downarrow{\scriptstyle FU\varepsilon(b)} & & \downarrow{\scriptstyle 1_a \varepsilon(b)} \\ FU1_B(b) & \xrightarrow{\varepsilon(b)} & 1_B \end{array}$$

that is

$$\begin{array}{ccc} FUFU(b) & \xrightarrow{\varepsilon FU(b)} & FU(b) \\ \downarrow{\scriptstyle FU\varepsilon(b)} & & \downarrow{\scriptstyle \varepsilon(b)} \\ FU(b) & \xrightarrow{\varepsilon(b)} & b \end{array}$$

$\varepsilon(b) : b \to \varepsilon(b)$ arrow of B

$\varepsilon : FU \to 1_B$

$\varepsilon(b) : FU(b) \to b$

$$b \xrightarrow{U} U(b) \xrightarrow{F} FU(b) \xrightarrow{\varepsilon(b)} b$$

replace f by $\varepsilon(b)$, a by $FU(b)$ in naturality $\varepsilon(b)FU(f)a = f\varepsilon(a)a$
$\varepsilon(b)FU(\varepsilon(b))FU(b) = \varepsilon\varepsilon(FU(b))FU(b)$
remove $FU(b)$ on right,
$\varepsilon(b)FU(\varepsilon(b)) = \varepsilon(b)\varepsilon(FU(b))$
this shows commutativity of previous diagram
$\varepsilon(b)\varepsilon(FU(b)) = \varepsilon(b)FU(\varepsilon(b))$
that is
$\varepsilon\varepsilon FU = \varepsilon FU\varepsilon$

21 Yoneda Functor

$Y : A \to Sets^{A^{op}}$
$Hom_A : A^{op} \times A \to Sets$
$g : a' \to a, h : b \to b'$
$Hom_A((g,h)) : Home_A(a,b) \to \{hfg | f \in Home_A(a,b)\}$
$Hom_A((g,h) \circ (g',h') : Home_A(a,b) \to \{hh'fgg' | f \in Home_A(a,b)\}$
$Hom_A((g,h))Hom_A((g',h')) : Home_A(a,b) \to \{h'fg' | f \in Home_A(a,b)\} \to \{hh'fgg' | f \in Home_A(a,b)\}$

$$\begin{array}{ccccc} a & \xrightarrow{g'} & a' & \xrightarrow{g} & a'' \\ & & \downarrow & & \downarrow f \\ b & \xleftarrow{h'} & b' & \xleftarrow{h} & b'' \end{array}$$

$Hom_A^* : A^{op} \to Sets^A$
$f^{op} : a \to c(f : c \to a)$
$g^{op} : c \to d(g : d \to c)$
$Home_A^*(a) : a \to \lambda b.Hom_A(a,b)$
$Home_A^*(f^{op}) : (a \to \lambda b.Hom_A(a,b)) \to (c \to \lambda b.Hom_A(f(c),b))$
$Home_A^*(g^{op}f^{op}) : (a \to \lambda b.Hom_A(a,b)) \to (d \to \lambda b.Hom_A(fg(d),b))$
$Home_A^*(g^{op})Home_A^*(f^{op}) : (a \to \lambda b.Hom_A(a,b)) \to (c \to \lambda b.Hom_A(f(c),b)) \to (d \to \lambda b.Hom_A(fg(d),b))$

$Hom^*_{A^{op}} : A \to Sets^{A^{op}}$

$f : c \to b\ g : d \to c$

$Home^*_{A^{op}}(b) : b \to \lambda a.Hom_{A^{op}}(a,b)$

$Home^*_{A^{op}}(f) : (b \to \lambda a.Hom_{A^{op}}(a,b)) \to (c \to \lambda a.Hom_{A^{op}}(a,f(c)))$

$Home^*_{A^{op}}(gf) : (b \to \lambda a.Hom_{A^{op}}(a,b)) \to (d \to \lambda a.Hom_{A^{op}}(a,gf(d)))$

$Home^*_{A^{op}}(g)Home^*_{A^{op}}(f) : (b \to \lambda a.Hom_{A^{op}}(a,b)) \to (c \to \lambda a.Hom_{A^{op}}(a,f(c))) \to (d \to \lambda a.Hom_{A^{op}}(a,gf(d)))$

Arrows in $Set^{A^{op}}$?

$f : b \to c = (b \to \lambda a.Hom_{A^{op}}(a,b)) \to (c \to \lambda a.Hom_{A^{op}}(a,f(c)))$

$Set^{A^{op}} : A^{op} \to Set$

an object $b = \lambda a.Hom_{A^{op}}(a,b)$ is a functor from A^{op} to Set.

$t : (\lambda a.Hom_{A^{op}}(a,b)) \to (\lambda a.Hom_{A^{op}}(a,t(c)))$ should be a natural transformatin.

$f^{op} : (b : A^{op}) \to (c : A^{op}) = f : c \to b$

$$\begin{array}{ccc}
Hom_{A^{op}}(a,c) & \xrightarrow{t(c)} & Hom_{A^{op}}(a,t(c)) \\
\Big\downarrow {\scriptstyle Home^*A^{op}(a,f)} & & \Big\uparrow {\scriptstyle Home^*A^{op}(a,f)} \\
Hom_{A^{op}}(a,b) & \xrightarrow{t(b)} & Hom_{A^{op}}(a,t(b))
\end{array}$$

21.1 Contravariant functor

$h_a = Hom_A(-,a)$

$f : b \to c, Hom_A(f, 1_a) : Hom_A(c,a) \to Hom_A(b,a)$